Iowa City Math Circle Handouts

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1 Sequences and Series

1.1 Arithmetic Sequences

An *arithmetic sequence* is a sequence of real numbers in which the difference between consecutive terms is constant. This difference is known as the *common difference*. For example, the sequence

$$2, 4, 6, 8, 10, \ldots$$

is an example of an arithmetic sequence with common difference 2. We denote the common difference by d and the nth term of the sequence by a_n. Using the definition above, we have that for any arithmetic sequence $\{a_n\}_{n\geq 1}$, the nth term satisfies

$$a_n = a_1 + (n-1)d,$$

where n is a positive integer.

Let S_n be the sum of the first n terms of an arithmetic sequence $\{a_n\}_{n\geq 1}$. Then by the above observations, we have

$$S_n = a_1 + (a_1 + d) + \cdots + (a_1 + (n-1)d); \quad (1)$$
$$\text{and} \quad S_n = (a_1 + (n-1)d) + (a_1 + (n-2)d) + \cdots + a_1. \quad (2)$$

By adding the corresponding terms in equations (1) and (2), we get

$$\begin{aligned} 2S_n &= (2a_1 + (n-1)d) + (2a_1 + (n-1)d) + \cdots + (2a_1 + (n-1)d) \\ &= n(2a_1 + (n-1)d). \end{aligned}$$

Thus, S_n is given by

$$S_n = \frac{n(2a_1 + (n-1)d)}{2} = \frac{n(a_1 + a_n)}{2}.$$

This result should intuitively make sense; the sum of the first n terms of an arithmetic sequence is just n times the average of the first and the last terms.

1.1.1 Counting Terms in an Arithmetic Sequence

Example 1.1: How many terms are in the arithmetic sequence $8, 11, \ldots, 35$?

Solution: We begin our solution with the following question:

How many numbers are in the sequence $n, n+1, n+2, \ldots, m-1, m$? Equivalently, how many integers are between n and m, *inclusive*?

Intuitively, the answer seems to be $m - n$. However, this is not the case. This may seem puzzling, but as an example, let $n = 1$. How many numbers are in the sequence $1, 2 \ldots m$? It is obvious that it is just m. But if we had used the formula of $m-n$, we would have obtained $m-1$, which is wrong.

Instead, the answer is $m-n+1$. If we subtract $n-1$ from each term in the sequence $n, n+1, \ldots, m$, we obtain the new sequence $1, 2, \ldots, m-(n-1)$. How many terms are in this sequence? It is simply $m-(n-1) = m-n+1$.

Coming back to our original problem, we see that our sequence is not in the form $n, n+1, \dots m$. In other words, it is not made up of consecutive integers. However, we can transform this sequence into a sequence of consecutive integers.

First, we see that the common difference of this arithmetic sequence is 3. So dividing each term by 3 will give us a sequence with a common difference of 1. However, if we divide by 3, we get fractional terms instead of consecutive integers.

To remedy this problem, we notice that the integers in the sequence have a remainder of 2 when divided by 3, which motivates us to subtract 2 from each term in the sequence and subsequently divide by 3. Doing so, we get

$$8, 11, \dots, 32, 35 \quad \to \quad 6, 9, \dots, 30, 33 \quad \to \quad 2, 3, \dots, 10, 11.$$

Now, we have transformed the original sequence to a sequence of consecutive integers. Using the answer to the subquestion, we obtain that there are $11 - 2 + 1 = 10$ integers in the sequence.

1.2 Geometric Sequences

A *geometric sequence* is a sequence of real numbers in which the ratio of consecutive terms is a constant. This ratio is known as the *common ratio*, which we denote by r. The sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

is an example of a geometric sequence with common ratio $\frac{1}{2}$. Again, using the definition above, it follows that the nth term of a geometric sequence is given by

$$a_n = a_1 r^{n-1},$$

where n is a positive integer.

Let S_n denote the sum of the first n terms of a geometric sequence. Then

$$S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1} \tag{3}$$

and

$$rS_n = ra_1 + r^2 a_1 + \cdots + a_1 r^n. \tag{4}$$

By subtracting (4) from (3), we get

$$(1-r)S_n = a_1 - a_1 r^n.$$

Thus,

$$S_n = \frac{a_1 - a_1 r^n}{1-r} = a_1 \left(\frac{1-r^n}{1-r} \right).$$

1.3 Evaluating Infinite Geometric Series

A *series* is the sum of an infinite sequence of numbers. An example of an *infinite geometric series* would be

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

To find the sum of an infinite geometric sequence, we can use the above result for the sum of a finite geometric series.

Let the infinite geometric series have common ratio r and nth term a_n. Notice that if $|r| > 1$, as n approaches ∞, $|a_n|$ also approaches ∞. Thus, the sum of an infinite geometric series is well-defined and finite if and only if $|r| < 1$. Now, what happens when $|r| = 1$?

In the case that $|r| < 1$, as n approaches ∞, $a_1 r^{n-1}$ approaches zero. Therefore, the sum of an infinite geometric sequence with $|r| < 1$ is given by

$$S_\infty = \frac{a}{1-r}.$$

1.4 Arithmetico-Geometric Sequences

Let's start with an example.

Example 1.2: Find
$$\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots.$$

Solution: Let $S=\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots$. We can multiply S by 2 and subtract S from that to get a cancellation of terms.
$$2S=\frac{1}{2^0}+\frac{2}{2^1}+\frac{3}{2^2}+\cdots$$
$$S=2S-S=\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\cdots.$$

Using the formula for the sum of an infinite geometric series for the above expression, we get
$$S=\frac{\frac{1}{2^0}}{1-\frac{1}{2}}=2.$$

An *arithmetico-geometric sequence* is a sequence in which each term is the product of a term from an arithmetic sequence and a term from a geometric sequence. From this definition, the sequence from *Example 1.2* is an arithmetico-geometric sequence. To find the sum of any arithmetico-geometric sequence, such as
$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots,$$
where the a_i form an arithmetic sequence and the b_i form a geometric sequence, multiply the terms of the sequence by the ratio $\frac{b_2}{b_1}=\frac{b_3}{b_2}=\cdots$.

1.5 Telescopic Sums and Products

Let's introduce the technique with an example.

Example 1.3: Find
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}.$$

Solution: First, let's look at the kth term of this sum, which is $\frac{1}{k(k+1)}$. Notice that we have the relationship
$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}.$$

Substituting this identity into the sum and canceling like terms, we get
$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}&=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\&=\frac{1}{1}-\frac{1}{n+1}\\&=\frac{n}{n+1}.\end{aligned}$$

When trying to find the sum or product of a series of terms, a useful technique is to write each term in such a way that when you rewrite all terms, most of them cancel in some way. This technique is called *telescoping*, and can often help simplify sums and products of terms, especially ones that contain fractions, trigonometric functions, or factorials.

1.6 Exercises

Problem 1: (1986 AHSME Problem 9) The product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)$ equals
(A) $\frac{5}{12}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{11}{20}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{7}{10}$

Problem 2: (2004 AMC 12B Problem 8) A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?
(A) 5 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Problem 3: (2012 AIME 1 Problem 2) The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Problem 4: (1983 ARML) In an arithmetic progression, the ratio of the sum of the first r terms to the sum of the first s terms is $\frac{r^2}{s^2}$ $(r \neq s)$. Find the ratio of the 8th term to the 23rd term.

Problem 5: (MathCounts) In order, the first four terms of a sequence are $2, 6, 12$, and 72 where each term, beginning with the third term, is the product of the two preceding terms. If the ninth term is 2^a3^b, what is the value of $a+b$?

Problem 6: (2004 AMC 12A Problem 14) A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term in the geometric progression?
(A) 1 **(B)** 4 **(C)** 36 **(D)** 49 **(E)** 81

Problem 7: (2016 AIME I Problem 1) For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Problem 8: (2009 AMC 12A Problem 17) Let $a+ar_1+ar_1^2+ar_1^3+\cdots$ and $a+ar_2+ar_2^2+ar_2^3+\cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?
(A) 0 **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** $\frac{1+\sqrt{5}}{2}$ **(E)** 2

Problem 9: (2006 AMC 10A Problem 19) How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?
(A) 0 **(B)** 1 **(C)** 59 **(D)** 89 **(E)** 178

Problem 10: (2003 AIME II Problem 8) Find the eighth term of the sequence 1440, 1716, 1848, ..., whose terms are formed by multiplying the corresponding terms of two arithmetic sequences.

Problem 11: (Mathleague) Find

$$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1}.$$

Problem 12: (1982 USAMO Problem 2) Prove

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$