Iowa City Math Circle Handouts

Ananth Shyamal, Divya Shyamal, and Reece Yang

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4 Circle Properties

4.1 Arcs and Inscribed Angles

1. Inscribed Angle Theorem: The measure of an inscribed angle is half the measure of the arc it subtends.
2. Corollary: Any angle inscribed in a semicircle is a right angle.
3. Corollary: Two inscribed angles that subtend the same arc have equal measures.

4.2 Chords, Tangents, and Secants

A *chord* is a line segment with both of its endpoints on the same circle. A *secant* is a line that intersects a circle at two *distinct* points. A *tangent* is a line that intersects a circle at *exactly* one point.

There are some useful facts about chords, tangents, and secants, which could be useful on the exercises.

1. Two congruent chords have congruent intercepted arcs.
2. The perpendicular bisector of any chord on a circle passes through the center of the circle.
3. Two congruent chords are equidistant from the center of the circle they lie on.
4. A tangent line to a circle passing through point P on the circle is perpendicular to the radius of the circle passing through P.
5. Tangent segments to a circle from a point outside the circle are congruent.
6. Let chords $\overline{AB}$ and $\overline{CD}$ on a circle intersect at point E. Then
$$\angle AEC = \frac{1}{2}\left(\overset{\frown}{AC} + \overset{\frown}{BD}\right).$$
7. Let $\overline{BC}$ be a chord on a circle, and let $\overline{DC}$ be tangent to the circle at C such that it forms an acute angle with the chord. Then
$$\angle DCB = \frac{1}{2}\overset{\frown}{BC}.$$
8. The measure of an angle formed when two secants intersect at a point outside the circle is one-half the difference of the measures of the two intercepted arcs.

4.3 Power of a Point Theorem

The Power of a Point theorem has three different statements, corresponding to a circle with center O and radius r.

1. Let $\overline{AC}$ and $\overline{BD}$ be two distinct chords on a circle, and let their intersection point be E. Then
$$AE \cdot EC = BE \cdot ED = OE^2 - r^2.$$
2. Let $\overline{AB}$ be a tangent to a circle, where A is on the circle. Draw a secant to the circle from point B, intersecting the circle at points C and D (where C is closer to B than D). Then
$$AB^2 = BC \cdot BD = OB^2 - r^2.$$

3. Let points A, B, D, and E lie on a circle such that $\overline{AB}$ intersects $\overline{DE}$ at a unique point C outside the circle and $CA > CB$ and $CE > CD$. Then

$$CB \cdot CA = CD \cdot CE = OC^2 - r^2.$$

4.4 Cyclic Quadrilaterals

A *cyclic quadrilateral* is quadrilateral that can be inscribed in circle, i.e. its vertices lie on a circle. There are many useful properties of cyclic quadrilaterals.

1. Opposite angles of a cyclic quadrilateral always sum to 180°.

2. Drawing the diagonals of a cyclic quadrilateral and its circumscribed circle gives you that angles inscribed by the same side of the quadrilateral are congruent.

3. Let $ABCD$ be a cyclic quadrilateral, and let its diagonal intersect at E. Then

$$AE \cdot EC = BE \cdot ED,$$

a consequence of the Power of the Point theorem.

4. Ptolemy's Theorem: Let $ABCD$ be a cyclic quadrilateral. Then

$$AB \cdot CD + BC \cdot AD = BD \cdot AC.$$

5. Brahmagupta's Formula: The area of any cyclic quadrilateral with side lengths a, b, c, and d and semi-perimeter s is given

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Note that if a quadrilateral satisfies any one of the above results, then the quadrilateral is cyclic.

4.5 Exercises

Problem 1: (1960 AHSME Problem 32) Let $\overline{BC}$ and $\overline{DE}$ be two chords on a circle centered at point O that intersect at the center. Let A be a point on line $\overline{DE}$ such that $\overline{AB} \perp \overline{BC}$ and $\overline{AB}$ has a length twice the radius of the circle. Let P be a point on $\overline{AB}$ such that $AP = AD$, Then:
(A) $AP^2 = PB \times AB$ **(B)** $AP \times DO = PB \times AD$ **(C)** $AB^2 = AD \times DE$
(D) $AB \times AD = OB \times AO$ **(E)** none of these

Problem 2: (1963 AHSME Problem 32) Acute-angled $\triangle ABC$ is inscribed in a circle with center at O; $\overset{\frown}{AB} = 120°$ and $\overset{\frown}{BC} = 72°$. A point E is taken in minor arc AC such that OE is perpendicular to AC. Then the ratio of the magnitudes of $\angle OBE$ and $\angle BAC$ is:
(A) $\frac{5}{18}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{4}{9}$

Problem 3: (1967 AHSME Problem 29) Let $\overline{AB}$ be a diameter of a circle. Tangents $\overline{AD}$ and $\overline{BC}$ are drawn so that $\overline{AC}$ and $\overline{BD}$ intersect in a point on the circle. If $\overline{AD} = a$ and $\overline{BC} = b$, $a \neq b$, the diameter of the circle is:
(A) $|a - b|$ **(B)** $\frac{1}{2}(a + b)$ **(C)** $\sqrt{ab}$ **(D)** $\frac{ab}{a+b}$ **(E)** $\frac{1}{2}\frac{ab}{a+b}$

Problem 4: (AHSME 1967 Problem 32) In quadrilateral $ABCD$ with diagonals AC and BD, intersecting at O, $BO = 4$, $OD = 6$, $AO = 8$, $OC = 3$, and $AB = 6$. The length of AD is:
(A) 9 **(B)** 10 **(C)** $6\sqrt{3}$ **(D)** $8\sqrt{2}$ **(E)** $\sqrt{166}$

Problem 5: (1971 CMO Problem 1) DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the center of the circle. Join OE and extend OE to cut the circle at C. Given $EC = 1$, find the radius of the circle.

Problem 6: (2013 AMC 10A Problem 23) In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?
(A) 11 **(B)** 28 **(C)** 33 **(D)** 61 **(E)** 72

Problem 7: (ARML) In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

Problem 8: (ARML) Chords AB and CD of a given circle are perpendicular to each other and intersect at a right angle at point E. Given that $BE = 16$, $DE = 4$, and $AD = 5$, find CE.

Problem 9: (2004 AMC 10B Problem 24) In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of AD/CD?
(A) $\dfrac{9}{8}$ **(B)** $\dfrac{5}{3}$ **(C)** 2 **(D)** $\dfrac{17}{7}$ **(E)** $\dfrac{5}{2}$

Problem 10: A hexagon with sides of lengths 2, 2, 7, 7, 11, and 11 is inscribed in a circle. Find the diameter of the circle.

Problem 11: (2012 AMC 12A Problem 16) Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

Problem 12: (2017 AMC 12B Problem 18) The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?
(A) $\frac{120}{37}$ **(B)** $\frac{140}{39}$ **(C)** $\frac{145}{39}$ **(D)** $\frac{140}{37}$ **(E)** $\frac{120}{31}$