Iowa City Math Circle Handouts

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2 Stars and Bars and Its Applications

2.1 Stars and Bars for Positive Integers

Example 2.1: How any ordered triples (x, y, z) of *positive* integers exist such that $x + y + z = 5$?

Solution: If we try listing out all the possibilities for the ordered triple, we find that there are 6: $(1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1)$. However, there is a better way to solve this problem using Stars and Bars.

If we let a star represent a quantity of one, we can represent 5 as

$$* * * * *$$

Using two dividers, we can divide the 5 stars into three different quantities to represent the values of x, y, and z. For example,

$$* * | * * | *$$

would represent $(2, 2, 1)$. The number of ordered triples will therefore be the number of ways to place the dividers. Each of the dividers could go into the $5 - 1 = 4$ possible slots between the five stars, so the number of ways to arrange the two dividers is $\binom{5-1}{3-1} = 6$.

More generally, the equation $x_1 + x_2 + \cdots + x_k = n$, where k, n are positive integers, has $\binom{n-1}{k-1}$ positive integer solutions.

Now, let's slightly change the problem.

2.2 Stars and Bars for Nonnegative Integers

Example 2.2: How any ordered triples (x, y, z) of *nonnegative* integers exist such that $x+y+z = 5$?
Solution: We can use the same method as above, except this time two dividers can be placed next to each other to show 0. For example,

$$*|| * * * *$$

would represent $(1, 0, 4)$. In this case, there are now $5 + 3 - 1 = 7$ slots to place the dividers, as there are $5 + 1 = 6$ slots around the stars in addition to the $3 - 2 = 1$ ways to place the divider with another divider.

More generally, the equation $x_1 + x_2 + \cdots + x_k = n$ has $\binom{n+k-1}{k-1}$ nonnegative integer solutions (or equivalently, $\binom{n+k-1}{n}$).

2.3 Stars and Bars Application in Counting

In many problems it may be necessary to find the number of ways to place n indistinguishable items into k distinguishable "boxes." Both versions of Stars and Bars can still be applied in these problems.
Example 2.3: How many ways are there to give 7 apples to your 2 best friends if each friend wants at least two apples?
Solution: First, since each friend wants at least two apples, lets distribute two apples to each friend so regardless of how we distribute the other apples, they will be happy. So now, we have to distribute the remaining $7 - 2 \cdot 2 = 3$ apples to the friends. We can rephrase the remaining problem as follows: How many ordered pairs (x, y) of *nonnegative* integers exist such that $x + y = 3$? From our investigation above, we see that the answer to this subproblem is $\binom{3+2-1}{2-1} = 4$ ways. Thus, the number of ways to distribute the apples to the friends is 4.

2.4 Exercises

Problem 1: (2003 AMC 10A Problem 21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?
(A) 22 **(B)** 25 **(C)** 27 **(D)** 28 **(E)** 729

Problem 2: (Mock AIME 2005 Problem 2) Find the number of 7 digit positive integers that have the property that their digits are in increasing order.

Problem 3: How many ways can you order the integers from 1 to 9 (inclusive) such that no two multiples of 3 are adjacent?

Problem 4: (2016 AMC 10A Problem 20) For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?
(A) 9 **(B)** 14 **(C)** 16 **(D)** 17 **(E)** 19

Problem 5: (2018 AMC 10A Problem 11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where n is a positive integer. What is n?
(A) 42 **(B)** 49 **(C)** 56 **(D)** 63 **(E)** 84

Problem 6: (AoPS) How many six-degree polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 30$ and $f(-1) = 12$?

Problem 7: (Mandelbrot) 20 chairs are set in a row. 5 people randomly sit in the chairs. What is the probability that nobody is sitting next to anybody else?

Problem 8: (1998 AIME Problem 7) Find the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$.

Problem 9: (1984 AIME Problem 11) A gardener plants three maple trees, four oaks, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Find the probability that no two birch trees are next to one another.

Problem 10: (AoPS) In how many ways can 8 licorice sticks and 12 chocolate bars be distributed to 5 kids if each kid must receive at least one piece of candy, but no kids can receive both types of candy?

Problem 11: (Mandelbrot) In a certain lottery, 7 balls are drawn at random from n balls numbered from 1 through n. If the probability that no pair of consecutive numbers is drawn equals the probability of drawing exactly one pair of consecutive numbers, find n.

Problem 12: (2008 AIME I Problem 11) Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?